

Generic families of matrix pencils and their bifurcation diagrams

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Abstract

V. I. Arnold (“On matrices depending on parameters”, *Russian Math. Surveys* 26, no. 2, 1971, 29–43) constructed smooth generic families of matrices with respect to similarity transformations depending smoothly on the entries of matrices and got bifurcation diagrams of such families with a small number of parameters. We extend these results to pencils of matrices.

1 Introduction

V. I. Arnold [1] (see also [2]) obtained a miniversal deformation of a Jordan matrix; that is, a simplest possible canonical form, to which not only a given square matrix A , but also an arbitrary family of matrices close to A , can

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be reduced by means of a similarity transformation that depends smoothly on the entries of A . Using this miniversal deformation, Arnold ([1], [2]) constructed bifurcation diagrams for generic smooth one-, two- and three-parameter families of matrices $A(\alpha_1, \dots, \alpha_n)$, $n \leq 3$; that is, he described all possible types of Jordan forms of $A(\alpha_1, \dots, \alpha_n)$ in a neighborhood of $\vec{0}$. The results are important for applications in which one has matrices that arise from physical measurements, which means that their entries are known only approximately.

Miniversal deformations of matrix pencils were obtained in [4] and [8]. In this article we construct bifurcation diagrams for generic smooth zero-, one- and two-parameter families of pencils.

The case of zero parameter families is trivial. (Of course, “a zero parameter family of pencils” means “a pencil”.) To make our consideration clearer, we study this case twice: by usual methods in Theorem 1.1 and by Arnold’s method of bifurcation diagrams in Theorem 3.1.

The theory of matrix pencils is the theory of pairs of matrices of the same size up to equivalence. Two matrix pairs (A_1, A_2) and (B_1, B_2) are *equivalent* if there exist two nonsingular matrices R and S such that

$$B_1 = RA_1S, \quad B_2 = RA_2S. \quad (1)$$

As was proved by Kronecker (see [7]), every pair of complex $m \times n$ matrices is equivalent to a direct sum, determined uniquely up to permutation of summands, of pairs of the form

$$(I_r, J_r(\lambda)), (J_r(0), I_r), (F_r, K_r), (F_r^T, K_r^T), \quad (2)$$

where $J_r(\lambda)$ is the Jordan cell with units over the diagonal and

$$F_r = \begin{bmatrix} 1 & & & 0 \\ 0 & \ddots & & \\ & \ddots & 1 & \\ 0 & & & 0 \end{bmatrix}, \quad K_r = \begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & 0 & \\ 0 & & & 1 \end{bmatrix}$$

are matrices of size $r \times (r - 1)$, $r \geq 1$.

Every square matrix is transformed to a diagonalizable matrix by a small jiggling. This result is extended to matrix pencils in the next theorem (the same extension but in terms of bifurcation diagrams is given in Theorem 3.1).

Theorem 1.1. *Every pair (A, B) of complex $m \times n$ matrices is transformed by an arbitrarily small jiggling to a pair that is equivalent to the following pair:*

- (i) $(I_n, \text{diag}(\alpha_1, \dots, \alpha_n))$ if $m = n$,
- (ii) $([I \ 0], [0 \ I])$ if $m < n$, and
- (iii) $([I \ 0]^T, [0 \ I]^T)$ if $m > n$.

Proof. Let $m = n$. We make A nonsingular by an arbitrarily small perturbation, reduce it to I , then make B diagonalizable by an arbitrarily small perturbation.

Let $m \neq n$, suppose $m < n$ (otherwise, consider the pair (A^T, B^T)). We make the rows of A linearly independent by an arbitrarily small perturbation, then reduce it to the form $A = [I \ 0]$ and partition $B = [B_1 \ B_2]$ conformal with A .

If the number of rows of B_2 is no more than the number of columns, we make its rows linearly independent by an arbitrarily small perturbation, reduce B_2 to the form $[0 \ I]$ and make $B_1 = 0$ by column transformations. The pair (A, B) takes the form (ii).

If the number of rows of B_2 is more than the number of columns, we make the columns of B_2 linearly independent by an arbitrarily small perturbation, reduce B_2 to the form $[0 \ I]^T$, then reduce B to the form

$$B = \begin{bmatrix} B_{11} & 0 \\ 0 & I \end{bmatrix}$$

by additions of columns of B_2 . The corresponding horizontal division of A crosses the block I , so we make a vertical division and obtain

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \quad B = \begin{bmatrix} C_1 & C_2 & 0 \\ 0 & 0 & I \end{bmatrix}. \quad (3)$$

Applying the same transformations to the fragment $[I \ 0]$, $[C_1 \ C_2]$, we reduce (A, B) respectively to the form (ii) (making $C_2 = [0 \ I]$ and $C_1 = 0$) or to the form

$$A = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}, \quad B = \begin{bmatrix} D_1 & D_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

(these transformations spoil the reduced part of the pair (3), but it is recovered by obvious transformations). We repeat this reduction until obtain (A, B) of the form (ii). \square

Note that miniversal deformations and bifurcation diagrams for real matrices up to similarity were given by Galin [5]; for certain classes of operators in metric spaces in [3], [6], [9], [10], and [11].

2 Generic families of matrix pencils

We study families of pairs of complex $m \times n$ matrices $\mathcal{A}(\vec{\alpha}) = (A_1(\vec{\alpha}), A_2(\vec{\alpha}))$, $\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$, holomorphic at $\vec{0}$. The entries of $A_1(\vec{\alpha})$ and $A_2(\vec{\alpha})$ are power series of complex parameters $\alpha_1, \dots, \alpha_k$ that are convergent in a neighborhood of $\vec{0}$. (The germ of a family $\mathcal{A}(\vec{\alpha})$ at $\vec{0}$ is called a *deformation* of the pair $\mathcal{A}(\vec{0})$, see [1]–[2].)

Two families $\mathcal{A}(\vec{\alpha})$ and $\mathcal{B}(\vec{\alpha})$ are called *equivalent* if there exist matrices $R(\vec{\alpha})$ and $S(\vec{\alpha})$ holomorphic at $\vec{0}$ such that $R(\vec{0}) = I$, $S(\vec{0}) = I$, and

$$B_1(\vec{\alpha}) = R(\vec{\alpha})A_1(\vec{\alpha})S(\vec{\alpha}), \quad B_2(\vec{\alpha}) = R(\vec{\alpha})A_2(\vec{\alpha})S(\vec{\alpha})$$

in a neighborhood of $\vec{0}$ (compare with (1)). A family $\mathcal{A}(\alpha_1, \dots, \alpha_k)$ is called *versal* if every family $\mathcal{B}(\beta_1, \dots, \beta_l)$ with $\mathcal{B}(\vec{0}) = \mathcal{A}(\vec{0})$ is equivalent to a family $\mathcal{A}(\varphi_1(\vec{\beta}), \dots, \varphi_k(\vec{\beta}))$, where $\varphi_i(\vec{\beta})$, $\varphi_i(\vec{0}) = \vec{0}$, are power series convergent in a neighborhood of $\vec{0}$. A versal family with the minimum possible number k of parameters is said to be *miniversal*.

For every pair of $m \times n$ matrices (A_1, A_2) , a miniversal family $\mathcal{A}(\vec{\alpha})$ with $\mathcal{A}(\vec{0}) = (A_1, A_2)$ was obtained in [4] and simplified in [8]. We now recall the result of [8]. It suffices to construct a miniversal family for a Kronecker canonical pair

$$(A_1, A_2) = \bigoplus_{i=1}^l (F_{p_i}, K_{p_i}) \oplus (I, C) \oplus (D, I) \oplus \bigoplus_{i=1}^r (F_{q_i}^T, K_{q_i}^T) \quad (4)$$

(see (2)), where $p_1 \leq \dots \leq p_l$, $q_1 \geq \dots \geq q_r$,

$$C = \bigoplus_{i=1}^t \Phi_i(\lambda_i) \quad (\lambda_i \neq \lambda_j \text{ if } i \neq j), \quad D = \Phi_0(0), \quad (5)$$

and $\Phi_0(0), \Phi_1(\lambda_1), \dots, \Phi_t(\lambda_t)$ have the form

$$\Phi_i(\lambda_i) = \text{diag}(J_{s_{i1}}(\lambda_i), J_{s_{i2}}(\lambda_i), \dots), \quad s_{i1} \geq s_{i2} \geq \dots, \quad \lambda_0 := 0.$$

Denote by 0^\uparrow (resp., 0^\downarrow , 0^\leftarrow , 0^\rightarrow) a matrix, in which all entries are zero except for the entries of the first row (respectively, the last row, the first column, the last column) that are independent parameters; and denote by Z the $p \times q$ matrix, in which the first $\max\{q - p, 0\}$ entries of the first row are independent parameters and the other entries are zeros:

$$0^\uparrow = \begin{bmatrix} * & \cdots & * \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} * & \cdots & * & 0 & \cdots & 0 \\ & & & & \ddots & \\ & 0 & & 0 & \cdots & 0 \end{bmatrix},$$

where the stars denote independent parameters. Let

$$\mathcal{H} = [H_{ij}] \tag{6}$$

be a block matrix, whose $p_i \times q_j$ blocks H_{ij} are of the form $H_{ij} = 0^\leftarrow$ if $p_i \leq q_j$, and $H_{ij} = 0^\downarrow$ if $p_i > q_j$.

Theorem 2.1 (see [8]). *One of miniversal families with the pair (4) at $\vec{0}$ is $\mathcal{M}(\vec{\alpha}) =$*

$$\left(\begin{array}{c|c|c|c} \begin{array}{ccc} \overline{F}_{p_1} & & \\ & F_{p_2} & 0 \\ & 0 & \ddots \\ & & F_{p_l} \end{array} & \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0^\downarrow \end{array} & \begin{array}{c} 0^\downarrow \\ 0^\downarrow \\ \vdots \\ 0^\downarrow \end{array} & \begin{array}{c} 0^\rightarrow \\ 0^\rightarrow \\ \vdots \\ 0^\rightarrow \end{array} \\ \hline & I & 0 & 0 \\ \hline & & \tilde{D} & 0^\rightarrow 0^\rightarrow \cdots 0^\rightarrow \\ \hline 0 & & \begin{array}{ccc} F_{q_1}^T & & \\ & F_{q_2}^T & 0 \\ & & \ddots \\ 0 & & F_{q_r}^T \end{array} \end{array} \right), \quad \left(\begin{array}{c|c|c|c} \begin{array}{ccc} \overline{K}_{p_1} & Z & \cdots & Z \\ & K_{p_2} & \ddots & \vdots \\ & 0 & \ddots & Z \\ & & & K_{p_l} \end{array} & \begin{array}{c} 0^\uparrow \\ 0^\uparrow \\ \vdots \\ 0^\uparrow \end{array} & 0 & \begin{array}{c} 0^\uparrow \\ 0^\uparrow \\ \vdots \\ 0^\uparrow \end{array} \\ \hline & \tilde{C} & 0 & 0^\leftarrow 0^\leftarrow \cdots 0^\leftarrow \\ \hline & & I & 0 \\ \hline 0 & & \begin{array}{ccc} K_{q_1}^T & Z^T & \cdots & Z^T \\ & K_{q_2}^T & \ddots & \vdots \\ & & & Z^T \\ 0 & & & K_{q_r}^T \end{array} \end{array} \right),$$

where

$$\tilde{C} = \bigoplus_{i=1}^t (\Phi_i(\lambda_i) + \mathcal{H}_i), \quad \tilde{D} = \Phi_0(0) + \mathcal{H}_0$$

(see (5)), and \mathcal{H}_i is of the form (6).

Note that \tilde{C} and \tilde{D} are miniversal deformations of C and D under similarity, which were given by Arnold [1]–[2].

Extending Arnold's notation from [1]–[2], we will denote the pairs (2) by the symbols

$$\lambda^r := (I_r, J_r(\lambda)), \quad \infty^r := (J_r(0), I_r), \quad \Delta^r := (F_r, K_r), \quad \nabla^r := (F_r^T, K_r^T)$$

and a Kronecker canonical pair of matrices by a sequence of these symbols. The complex number λ and the symbol ∞ will be called the *eigenvalues* of $(I_r, J_r(\lambda))$ and $(J_r(0), I_r)$; the eigenvalues will be denoted by small Greek letters. We will say that two pairs of $m \times n$ matrices have the same *Kronecker type* if their Kronecker canonical forms differ only by the sets of distinct eigenvalues; the Kronecker type will be given by an unordered sequence of symbols $\Delta^r, \nabla^r, \lambda^r$ ($r \in \mathbb{N}$, $\lambda \in \mathbb{C} \cup \infty$, the set of λ 's in the sequence is determined up to bijections in $\mathbb{C} \cup \infty$). In particular, $(I_r, J_r(5))$ and $(J_r(0), I_r)$ have the same Kronecker type λ^r ; $(I_2, J_2(5))$, $(I_2, 5I_2)$, and $(I_2, \text{diag}(5, 6))$ have distinct Kronecker types λ^2 , $\lambda\lambda$, and $\lambda\mu$.

In the next section, we will study the set of Kronecker types of matrix pairs that form a miniversal family $\mathcal{M}(\vec{\alpha})$ from Theorem 2.1 in a neighborhood of $\vec{0}$. In this section, we remove parameters that have no effect on the Kronecker type. Let us denote by

$$\mathcal{M}'(\vec{\beta}) \tag{7}$$

the family that is obtained from $\mathcal{M}(\vec{\alpha})$ by replacement of its blocks \tilde{C} and \tilde{D} with

$$\tilde{C}' = \bigoplus_i (\Phi_i(\lambda_i) + \mathcal{H}'_i) \quad \text{and} \quad \tilde{D}' = \Phi_0(0) + \mathcal{H}'_0,$$

where \mathcal{H}'_i is obtained from \mathcal{H}_i by replacement of its upper-left-hand entry with 0 (the number of parameters decreases by the number of distinct eigenvalues). Furthermore, denote by

$$\mathcal{M}''(\vec{\beta}, \vec{\gamma}) \tag{8}$$

the family that is obtained from $\mathcal{M}'(\vec{\beta})$ by replacement of \tilde{C}' with

$$\tilde{C}'' = \bigoplus_i (\Phi_i(\lambda_i) + \mathcal{H}'_i + \gamma_i I) = \bigoplus_i (\Phi_i(\lambda_i + \gamma_i) + \mathcal{H}'_i)$$

and \tilde{D}' with

$$\tilde{D}'' = \Phi_0(0) + \mathcal{H}'_0 + \gamma_0 I = \Phi_0(\gamma_0) + \mathcal{H}'_0.$$

The families $\mathcal{M}''(\vec{\beta}, \vec{\gamma})$ and $\mathcal{M}(\vec{\alpha})$ have the same number of parameters. Moreover, $\mathcal{M}''(\vec{\beta}, \vec{\gamma})$ is a miniversal family too; this fact is proved in the same way as the miniversality of $\mathcal{M}(\vec{\alpha})$ (Theorem 2.1 from [8]) and is based on the following criterion.

A criterion of miniversality (see [1]–[2] and [8]): A family

$$\mathcal{A}(\alpha_1, \dots, \alpha_t) = \mathcal{A}_0 + \sum_{i=1}^t \alpha_i \mathcal{A}_i + \dots, \quad \mathcal{A}_i = (A_i, B_i) \in \mathbb{C}^{(m \times n, m \times n)}, \quad (9)$$

(where $\mathbb{C}^{(m \times n, m \times n)}$ is the vector space of pairs of complex $m \times n$ matrices, and the points after $+$ denote the terms of order more than 1) is miniversal if and only if

$$\mathbb{C}^{(m \times n, m \times n)} = \mathcal{P}_{\mathcal{A}} \oplus \mathcal{T}_{\mathcal{A}},$$

where

$$\mathcal{P}_{\mathcal{A}} = \{\alpha_1 \mathcal{A}_1 + \dots + \alpha_t \mathcal{A}_t \mid \alpha_i \in \mathbb{C}\} \quad (10)$$

is the vector space spanned by $\mathcal{A}_1, \dots, \mathcal{A}_t$, and

$$\mathcal{T}_{\mathcal{A}} = \{(RA_0 - A_0S, RB_0 - B_0S) \mid R \in \mathbb{C}^{m \times m}, S \in \mathbb{C}^{n \times n}\}$$

is the tangent space at the point \mathcal{A}_0 to the equivalence class

$$\{(RA_0S, RB_0S) \mid R \in \text{Gl}_m(\mathbb{C}), S \in \text{Gl}_n(\mathbb{C})\}$$

of the pair \mathcal{A}_0 .

Similar to [1]–[2], we say that a family (9) is *transversal to the stratification into Kronecker types* if

$$\mathbb{C}^{(m \times n, m \times n)} = \mathcal{P}_{\mathcal{A}} + \mathcal{Q}_{\mathcal{A}}, \quad (11)$$

where $\mathcal{P}_{\mathcal{A}}$ is defined by (10) and $\mathcal{Q}_{\mathcal{A}}$ is the tangent space at the point \mathcal{A}_0 to the class of all pairs of matrices having the same Kronecker type as \mathcal{A}_0 . (Two subspaces are *transversal* if their sum is the entire space.)

Theorem 2.2. (i) *In the space of families of pairs of $m \times n$ matrices, the families transversal to the stratification into Kronecker types constitute an everywhere dense set.*

(ii) *For every miniversal family $\mathcal{M}(\vec{\alpha})$ from Theorem 2.1, the family $\mathcal{M}'(\vec{\beta})$ (see (7)) is transversal to the stratification into Kronecker types. Moreover, the corresponding sum (11) is direct:*

$$\mathbb{C}^{(m \times n, m \times n)} = \mathcal{P}_{\mathcal{M}'} \oplus \mathcal{Q}_{\mathcal{M}'}. \quad (12)$$

Proof. The statement (i) follows from the theorem of [2, § 30E]. Let us prove the statement (ii). The family (8) has the form

$$\mathcal{M}''(\vec{\beta}, \vec{\gamma}) = \mathcal{M}'(\vec{\beta}) + \gamma_1 \mathcal{M}_1'' + \cdots + \gamma_s \mathcal{M}_s''$$

and has the same Kronecker type as $\mathcal{M}'(\vec{\beta})$ for a small $\vec{\gamma}$, therefore, $\gamma_1 \mathcal{M}_1'' + \cdots + \gamma_s \mathcal{M}_s'' \in \mathcal{Q}_{\mathcal{M}'}$. Since $\mathcal{M}''(\vec{\beta}, \vec{\gamma})$ is a miniversal family,

$$\begin{aligned} \mathbb{C}^{(m \times n, m \times n)} &= \mathcal{P}_{\mathcal{M}''} \oplus \mathcal{T}_{\mathcal{M}''} \\ &= \mathcal{P}_{\mathcal{M}'} \oplus \{\gamma_1 \mathcal{M}_1'' + \cdots + \gamma_s \mathcal{M}_s'' \mid \gamma_i \in \mathbb{C}\} \oplus \mathcal{T}_{\mathcal{M}''} \\ &= \mathcal{P}_{\mathcal{M}'} \oplus \mathcal{Q}_{\mathcal{M}'}, \end{aligned}$$

this proves (12). □

Similar to [1]–[2], a family will be called a *generic family* (or a family in general position) if it is transversal to the stratification into Kronecker types.

Corollary 2.1. *A nongeneric family can be transformed into a generic family by an arbitrarily small perturbation of the family. Since the sum (12) is a direct sum, the families $\mathcal{M}'(\vec{\beta})$ have the most complicated Kronecker structure among the generic families: if an arbitrary family $\mathcal{A}(\vec{\delta})$ has the same number of parameters as $\mathcal{M}'(\vec{\beta})$ but contains matrices with more complicated Kronecker structure, then $\mathcal{A}(\vec{\delta})$ is not a generic family.*

3 Bifurcation diagrams

In this section, we construct bifurcation diagrams for generic zero-, one- and two-parameter families of pairs of matrices.

Let $\mathcal{A}(\vec{\alpha})$ be a family of pairs of $m \times n$ matrices; that is, a holomorphic mapping

$$\mathcal{A} : \Lambda \rightarrow \mathbb{C}^{(m \times n, m \times n)},$$

where $\Lambda \subset \mathbb{C}^k$ is a neighborhood of $\vec{0}$. A *bifurcation diagram* of this family is, by definition, a partition of the parameter domain Λ according to Kronecker types of pairs. To construct it we assign to each $\vec{\alpha} \in \Lambda$ the Kronecker type of $\mathcal{A}(\vec{\alpha})$ and then join all points with the same Kronecker type. We narrow down the neighborhood Λ when it simplifies the structure of the bifurcation diagram. The bifurcation diagram of a generic family reflects the possible Kronecker structure of pairs in the family. By [2, §30E], if in the study of a phenomenon we obtain another bifurcation diagram then in the idealization of the phenomenon something essential was missed, or there were some special reasons for an additional complexity of the structure, or the family is not generic.

If all pairs $\mathcal{A}(\vec{\alpha})$, $\vec{\alpha} \in \Lambda$, have the same Kronecker type \bar{t} , we will give the bifurcation diagram by the sequence \bar{t} . But usually matrix pairs of a generic family have distinct types at $\vec{0}$ and outside of $\vec{0}$; in this case we will give the bifurcation diagram by the pair \bar{t}_0/\bar{t}_1 , where \bar{t}_0 is the Kronecker type of $\mathcal{A}(\vec{0})$ and \bar{t}_1 is the set of Kronecker types of $\mathcal{A}(\vec{\alpha})$, $\vec{0} \neq \vec{\alpha} \in \Lambda$.

Theorem 3.1. *Generic zero-parameter families of pairs of $m \times n$ matrices have the bifurcation diagrams*

$$\left. \begin{array}{l} \Delta^r \dots \Delta^r \Delta^{r+1} \dots \Delta^{r+1} \quad \nabla^r \dots \nabla^r \nabla^{r+1} \dots \nabla^{r+1} \\ \lambda \mu \dots \tau \end{array} \right\} \quad (13)$$

(the parts $\Delta^{r+1} \dots \Delta^{r+1}$ and $\nabla^{r+1} \dots \nabla^{r+1}$ can be absent).

Proof. Let \mathcal{A} be a generic zero-parameter family. By Corollary 2.1, we may suppose that $\mathcal{A} = \mathcal{M}'(\vec{\beta})$, where $\mathcal{M}(\vec{\alpha})$ is a family from Theorem 2.1. Selecting all $\mathcal{M}(\vec{\alpha})$ for which $\mathcal{M}'(\vec{\beta})$ is a zero-parameter family, we obtain the list (13) of all possible Kronecker types for \mathcal{A} . (Note that the pairs (i)–(iii) from Theorem 1.1 have the types (13); this gives another proof of Theorem 1.1.) \square

Theorem 3.2. *Generic one-parameter families of pairs of $m \times n$ matrices have the bifurcation diagrams (13), in which case the families behave as generic zero-parameter families, or have the following bifurcation diagrams:*

$$\left. \begin{array}{l} \Delta^r \Delta^{r+2} / \Delta^{r+1} \Delta^{r+1} \quad \nabla^r \nabla^{r+2} / \nabla^{r+1} \nabla^{r+1} \\ \Delta^r \lambda / \Delta^{r+1} \quad \nabla^r \lambda / \nabla^{r+1} \\ \lambda_1^2 \lambda_2 \dots \lambda_t / \mu_1 \mu_2 \dots \mu_{t+1} \end{array} \right\} \quad (14)$$

If in a one-parameter family there are pairs with a more complicated Kronecker structure, then we can remove them by an arbitrarily small perturbation of the family.

Proof. Let $\mathcal{A}(\beta)$ be a generic one-parameter family. If (11) is not a direct sum, then the parameter β does not affect on the Kronecker type and all pairs of the family have the same type. Applying Theorem 3.1, we get the list (13) of admissible bifurcation diagrams for the family.

Suppose (11) is a direct sum. By Corollary 2.1, we may take $\mathcal{A}(\beta) = \mathcal{M}'(\beta)$, where $\mathcal{M}(\vec{\alpha})$ is a family from Theorem 2.1. Selecting all $\mathcal{M}(\vec{\alpha})$ for which $\mathcal{M}'(\vec{\beta})$ is a one-parameter family, we obtain that $\mathcal{A}(0) = \mathcal{M}'(0)$ is one of the pairs:

$$\Delta^r \Delta^{r+2}, \quad \nabla^r \nabla^{r+2}, \quad \Delta^r \lambda, \quad \nabla^r \lambda, \quad \lambda_1^2 \lambda_2 \dots \lambda_t.$$

1) Let $\mathcal{A}(0) = \Delta^r \Delta^{r+2} = (F_r, K_r) \oplus (F_{r+2}, K_{r+2})$ (see (2)). Then $\mathcal{A}(\beta)$ is the pair of matrices

$$\begin{array}{c} \begin{array}{ccccc} & 1 & \dots & r-1 & \overline{1} & \dots & \overline{r+1} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ r \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{r+2} \end{array} & \begin{array}{|c|} \hline 1 \\ \hline \end{array} & & & \begin{array}{|c|} \hline 0 \\ \hline \end{array} \\ \hline & \begin{array}{|c|} \hline 0 \\ \hline \end{array} & & & \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} & 1 & \dots & r-1 & \overline{1} & \dots & \overline{r+1} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ r \\ \overline{1} \\ \overline{2} \\ \vdots \\ \overline{r+2} \end{array} & \begin{array}{|c|} \hline 0 \\ \hline \end{array} & & & \begin{array}{|c|} \hline \beta \\ \hline \end{array} \\ \hline & \begin{array}{|c|} \hline 0 \\ \hline \end{array} & & & \begin{array}{|c|} \hline 0 \\ \hline \end{array} \end{array} \end{array}$$

(we always number the rows and columns of the second summand by overbarred natural numbers). Rearranging the rows and columns in the order

$$(\overline{1}, \overline{2} | 1, \overline{3} | 2, \overline{4} | \dots | r-1, \overline{r+1} | r, \overline{r+2})$$

and

$$(\overline{1}, \overline{2} | 1, \overline{3} | 2, \overline{4} | \dots | r-2, \overline{r} | r-1, \overline{r+1})$$

(the symbol $|$ denotes the partition into strips), we obtain the following pair of matrices:

	$\overline{1}$	$\overline{2}$	1	$\overline{3}$	\cdots	$r-1$	$\overline{r+1}$
$\overline{1}$	1						
$\overline{2}$		1					
1			1				
$\overline{3}$				1			
\vdots					\ddots		
$r-1$						1	
$\overline{r+1}$							1
r							
$\overline{r+2}$							

	$\overline{1}$	$\overline{2}$	\cdots	$r-2$	\overline{r}	$r-1$	$\overline{r+1}$
$\overline{1}$	0	0					
$\overline{2}$	1	0					
1	β	0					
$\overline{3}$	0	1					
\vdots			\ddots				
$r-1$				1			
$\overline{r+1}$					1		
r						1	
$\overline{r+2}$							1

We may reduce this pair by simultaneous elementary transformations. Let us prove that a linear combination of rows of the (2,1) block of the second matrix may be added to rows of its (1,1) block without spoiling the other blocks of the pair. Indeed, additions of rows of the second horizontal strip to rows of the first horizontal strip spoil the (1,2) block of the first matrix. We recover it by additions of columns of the first vertical strip to columns of the second vertical strip, which spoil the (1,2) and (2,2) blocks of the second matrix. We recover them by additions of rows of the third horizontal strip spoiling the (1,3) and (2,3) blocks of the first matrix. These blocks are recovered by additions of columns of the first and the second vertical strips, and so on. On the last step, we recover blocks of the last vertical strip of the second matrix by additions of rows of the last horizontal strip without spoiling the other blocks since the last horizontal strip of the first matrix is zero.

Let the parameter $\beta \neq 0$. Multiplying the third rows by β^{-1} , we make unit the (3,1) entry of the second matrix spoiling the (3,3) entry of the first matrix. We recover it multiplying the third columns by β and so on until obtain the initial pair with $\beta = 1$. The subtraction of the first row of the (2,1) block of the second matrix from the second row of its (1,1) block makes zero this block. Up to simultaneous permutation of rows and columns,

the obtained pair has the form $(F_{r+1}, K_{r+1}) \oplus (F_{r+1}, K_{r+1})$. Therefore, the bifurcation diagram of $\mathcal{A}(\beta)$ is $\Delta^r \Delta^{r+2} / \Delta^{r+1} \Delta^{r+1}$.

2) The case $\mathcal{A}(0) = \nabla^r \nabla^{r+2}$ is considered analogously.

3) Let $\mathcal{A}(0) = \Delta^r \lambda$. First, suppose $\lambda \neq \infty$. Up to simultaneous permutation of rows and columns, $\mathcal{A}(\beta)$ is the pair

1			
	1		
	0	\ddots	
		\ddots	1
			0

λ			
β	0		
	1	\ddots	
		\ddots	0
			1

Analogous to the case 1), we make $\beta = 1$ in the second matrix, then we make $\lambda = 0$ in this matrix by adding β . The obtained pair is (F_{r+1}, K_{r+1}) .

Let now $\lambda = \infty$. Then $\mathcal{A}(\beta)$ is the pair

1			
0	\ddots		
	\ddots	1	
		0	β
			0

0			
1	\ddots		
	\ddots	0	
		1	
			1

Making $\beta = 1$ gives the pair (F_{r+1}, K_{r+1}) .

Therefore, the bifurcation diagram of $\mathcal{A}(\beta)$ is $\Delta^r \lambda / \Delta^{r+1}$.

4) The case $\mathcal{A}(0) = \nabla^r \lambda$ is considered analogously.

5) Let $\mathcal{A}(0) = \lambda_1^2 \lambda_2 \dots \lambda_t$. First, we suppose $\lambda_1 \neq \infty$. Then the pair $\mathcal{A}(\beta)$ has a direct summand

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 1 \\ \beta & \lambda_1 \end{bmatrix} \right).$$

This pair reduces to $(1, \mu_1) \oplus (1, \mu_2)$ if $\beta \neq 0$. The case $\lambda_1 = \infty$ is considered similarly. The bifurcation diagram of $\mathcal{A}(\beta)$ is $\lambda_1^2 \lambda_2 \dots \lambda_t / \mu_1 \mu_2 \dots \mu_{t+1}$. \square

Theorem 3.3. *Generic two-parameter families of pairs of $m \times n$ matrices have the bifurcation diagrams (13) and (14), in which case the families behave as generic zero-parameter and one-parameter families, or have the following bifurcation diagrams:*

- (i) $\Delta^1 \nabla^1 / \lambda$,
- (ii) $\Delta^r \Delta^{r+3} / \Delta^{r+1} \Delta^{r+2}$,
- (iii) $\Delta^r \Delta^r \Delta^{r+2} / \Delta^r \Delta^{r+1} \Delta^{r+1}$,
- (iv) $\Delta^r \Delta^{r+2} \Delta^{r+2} / \Delta^{r+1} \Delta^{r+1} \Delta^{r+2}$,
- (v) $\Delta^r \Delta^r \lambda / \Delta^r \Delta^{r+1}$,
- (vi) $\Delta^r \Delta^{r+1} \lambda / \{\Delta^r \Delta^{r+2} \text{ (this type have the pairs with parameters on a smooth line through } \vec{0}), \Delta^{r+1} \Delta^{r+1} \text{ (the pairs with parameters outside the line)}\}$,
- (vii) $\Delta^r \lambda \mu / \{\Delta^{r+1} \nu \text{ (the pairs with parameters on two smooth lines intersecting at } \vec{0}), \Delta^{r+2} \text{ (the pairs with parameters outside the lines)}\}$,
- (viii) $\lambda_1^3 \lambda_2 \dots \lambda_t / \{\mu_1^2 \mu_2 \dots \mu_{t+1} \text{ (the pairs with parameters on a line with a cusp at } \vec{0}), \mu_1 \mu_2 \dots \mu_{t+2} \text{ (the pairs with parameters outside the line)}\}$,
- (ix) $\lambda_1^2 \lambda_2^2 \lambda_3 \dots \lambda_t / \{\mu_1^2 \mu_2 \dots \mu_{t+1} \text{ (the pairs with parameters on two smooth lines intersecting at } \vec{0}), \mu_1 \mu_2 \dots \mu_{t+2} \text{ (the pairs with parameters outside the lines)}\}$,
- (x) the diagrams that are obtained from the diagrams (ii)–(vii) by replacing all symbols Δ by ∇ .

If in a two-parameter family there are pairs with a more complicated Kronecker structure, then we can remove them by an arbitrarily small perturbation of the family.

Proof. Let $\mathcal{A}(\beta, \gamma)$ be a generic two-parameter family. If (11) is not a direct sum, then the family behaves as a one-parameter family, so its bifurcation diagram is contained in the lists (13) and (14).

Let (11) be a direct sum, then we may take $\mathcal{A}(\beta, \gamma) = \mathcal{M}'(\beta, \gamma)$, where $\mathcal{M}(\vec{\alpha})$ is a family from Theorem 2.1. Selecting all $\mathcal{M}(\vec{\alpha})$ for which $\mathcal{M}'(\vec{\beta})$ is a two-parameter family, we obtain that $\mathcal{A}(0, 0) = \mathcal{M}'(0, 0)$ is one of the pairs:

$$\left. \begin{array}{lll} \Delta^1 \nabla^1 & \Delta^r \Delta^{r+3} & \Delta^r \Delta^r \Delta^{r+2} \\ \Delta^r \Delta^{r+2} \Delta^{r+2} & \Delta^r \Delta^r \lambda & \Delta^r \Delta^{r+1} \lambda \\ \Delta^r \lambda \mu & \lambda_1^3 \lambda_2 \dots \lambda_t & \lambda_1^2 \lambda_2^2 \lambda_3 \dots \lambda_t \end{array} \right\} \quad (15)$$

or it is obtained from them by turnover of all Δ and ∇ . We will consider only the pairs (15), the others are reduced to them by taking the transposed matrices.

1) Let $\mathcal{A}(0,0) = \Delta^1 \nabla^1 = (0,0)$. Then $\mathcal{A}(\beta, \gamma) = (\beta, \gamma)$; we have the bifurcation diagram (i).

2) Let $\mathcal{A}(0,0) = \Delta^r \nabla^{r+3}$. Let the rows and columns of $\mathcal{A}(\beta, \gamma)$ be numbered by $1, 2, \dots, r; \overline{1}, \overline{2}, \dots, \overline{r+3}$ and $1, 2, \dots, r-1; \overline{1}, \overline{2}, \dots, \overline{r+2}$. Rearranging them in the order

$$(\overline{1}, \overline{2}, \overline{3} | 1, \overline{4} | 2, \overline{5} | \dots | r-1, \overline{r+2} | r, \overline{r+3})$$

and

$$(\overline{1}, \overline{2}, \overline{3} | 1, \overline{4} | 2, \overline{5} | \dots | r-2, \overline{r+1} | r-1, \overline{r+2}),$$

we obtain the following pair of matrices:

1				0	0	0			
1				1	0	0			
1				0	1	0			
	1			β	γ	0			
	1			0	0	1			
		\ddots					\ddots		
			1					1	
			1					1	
								1	1

We will reduce the (1,1) and (2,1) blocks of the second matrix to the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (16)$$

preserving the other blocks. Similar to the case 1) of the proof of Theorem 3.2, we may add rows of the (2,1) block to rows of the (1,1) block.

Suppose $\gamma \neq 0$ (the case $\gamma = 0$ and $\beta \neq 0$ is simpler). Adding the second column to the first (and making the inverse transformations with rows to preserve the first matrix), we make the entry $\beta = 0$. Then we make zero the second column of the (1,1) block.

At last, we interchange the second and third rows to obtain the blocks (16). To preserve the form of the other blocks, we make the same permutation of columns, then we interchange the rows and columns within all strips except for the first horizontal and vertical strips.

Up to simultaneous permutation of rows and columns, the obtained pair has the form $\Delta^{r+1}\Delta^{r+2}$; we have the bifurcation diagram (ii).

3) Let $\mathcal{A}(0,0) = \Delta^r \Delta^r \Delta^{r+2}$. Rearranging the rows and columns of $\mathcal{A}(\beta, \gamma)$ in the order

$$(\bar{1}, \bar{2} | 1, \bar{1}, \bar{3} | \dots | r, \bar{r}, \overline{\overline{r+2}}) \quad \text{and} \quad (\bar{1}, \bar{2} | 1, \bar{1}, \bar{3} | \dots | r-1, \overline{\overline{r-1}}, \overline{\overline{r+1}}),$$

we obtain the following pair of matrices:

1				0	0			
1				1	0			
	1			β	0			
	1			γ	0			
		1		0	1			
			\ddots			\ddots		
							I_3	
								I_3

If $\beta \neq 0$ or $\gamma \neq 0$, then we reduce the (1,1) and (2,1) blocks of the second matrix to the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

preserving the other blocks. Up to simultaneous permutation of rows and columns, the obtained pair has the form $\Delta^r \Delta^{r+1} \Delta^{r+1}$; we have the bifurcation diagram (iii).

4) Let $\mathcal{A}(0,0) = \Delta^r \Delta^{r+2} \Delta^{r+2}$. We rearrange the rows and columns of $\mathcal{A}(\beta, \gamma)$ in the order

$$(\bar{1}, \bar{1} | \bar{2}, \bar{2} | 1, \bar{3}, \bar{3} | \dots | r, \overline{\overline{r+2}}, \overline{\overline{r+2}})$$

and

$$(\bar{1}, \bar{1} | \bar{2}, \bar{2} | 1, \bar{3}, \bar{3} | \dots | r-1, \overline{\overline{r+1}}, \overline{\overline{r+1}})$$

and obtain the following pair of matrices:

1					0	0				
	1				0	0				
		1			1					
			1			β	γ	1		
				\ddots					\ddots	
										I_3

If $\beta \neq 0$ or $\gamma \neq 0$, then we reduce the (1,1), (2,1), and (3,1) blocks of the second matrix to the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

preserving the other blocks. We have the bifurcation diagram (iv).

5) Let $\mathcal{A}(0,0) = \Delta^r \Delta^r \lambda$. Suppose $\lambda \neq \infty$ (the case $\lambda = \infty$ is considered analogously; it may be also reduced to the considered case by interchanging the matrices). We rearrange the rows and columns of $\mathcal{A}(\beta, \gamma)$ in the following manner:

$$(\bar{1} | 1, \bar{1} | \dots | r, \bar{r}) \quad \text{and} \quad (\bar{1} | 1, \bar{1} | \dots | r-1, \overline{r-1}).$$

The obtained pair of matrices is

1					λ				
	1				β				
		1			γ				
			I_2			I_2			
				\ddots			\ddots		
								I_2	
									I_2

Let $\beta \neq 0$ or $\gamma \neq 0$. We reduce the block $[\beta \ \gamma]^T$ to the form $[0 \ 1]^T$ by row transformations within the second horizontal strip (to preserve the form

of matrices, we make the same transformations within all horizontal strips except for the first strip, then the inverse transformations within the vertical strips except for the first strip). At last, we make the entry $\lambda = 0$. We have the bifurcation diagram (v).

6) Let $\mathcal{A}(0, 0) = \Delta^r \Delta^{r+1} \lambda$. Suppose $\lambda \neq \infty$ (the case $\lambda = \infty$ is considered analogously). We rearrange the rows and columns of $\mathcal{A}(\beta, \gamma)$ in the following manner:

$$(\bar{1} | \bar{1} | 1, \bar{2} | \dots | r, \overline{r+1}) \quad \text{and} \quad (\bar{1} | \bar{1} | 1, \bar{2} | \dots | r-1, \bar{r}).$$

The obtained pair of matrices is

1					
	1				
		1			
			1		
				I_2	
					\ddots
					I_2

λ					
γ					
β					
	1				
		I_2			
			\ddots		
				I_2	
					I_2

If $\beta \neq 0$, then we make the entries $\beta = 1$ and $\gamma = \lambda = 0$; the obtained pair is of type $\Delta^{r+1} \Delta^{r+1}$. If $\beta = 0$ and $\gamma \neq 0$, then we make the entries $\gamma = 1$ and $\lambda = 0$; the obtained pair is of type $\Delta^r \Delta^{r+2}$. We have the bifurcation diagram (vi).

7) Let $\mathcal{A}(0, 0) = \Delta^r \lambda \mu$. Consider the case $\lambda \neq \infty$ and $\mu \neq \infty$. Rearranging the rows and columns of $\mathcal{A}(\beta, \gamma)$ in the order

$$(\bar{1}, \bar{1} | 1, 2, \dots, r) \quad \text{and} \quad (\bar{1}, \bar{1} | 1, 2, \dots, r-1),$$

we obtain the following pair of matrices:

1	
	1
	1
	0 \ddots
	\ddots 1
	0

λ	
μ	
β γ	0
	1 \ddots
	\ddots 0
	1

If $\beta \neq 0$ and $\gamma \neq 0$, then we make the entries $(\beta, \gamma) = (0, 1)$ by column transformations within the first vertical strip (and by the inverse transformations within the first horizontal strip to preserve the first matrix). Since $\lambda \neq \mu$, the (1,1) block takes the form

$$\begin{bmatrix} \lambda & 0 \\ a & \mu \end{bmatrix}, \quad a \neq 0.$$

Adding the entry γ , we make the entry $\mu = 0$; adding a , we make $\lambda = 0$; then we make $a = 1$. The obtained pair is Δ^{r+2} .

If $\beta = 0$ and $\gamma \neq 0$, then we make $\gamma = 1$ and $\mu = 0$ and obtain the pair $\lambda \Delta^{r+1}$. If $\beta \neq 0$ and $\gamma = 0$, then the pair is reduced to $\mu \Delta^{r+1}$. We have the bifurcation diagram (vii).

8) Let $\mathcal{A}(0, 0) = \lambda_1^3 \lambda_2 \dots \lambda_t$. Consider the case $\lambda_1 \neq \infty$. Then $\mathcal{A}(\beta, \gamma)$ has a direct summand (I_3, A) , where

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ \beta & \lambda_1 & 1 \\ \gamma & 0 & \lambda_1 \end{bmatrix};$$

we may reduce A by similarity transformations. The characteristic polynomial of A is $\chi(x) = x^3 - \beta x - \gamma$, the roots of its derivative $\chi'(x) = 3x^2 - \beta$ are $\pm\sqrt{\beta/3}$. The matrix A has multiply eigenvalues if and only if $\chi(x)$ and $\chi'(x)$ have a common root; that is,

$$\pm\frac{\beta}{3}\sqrt{\frac{\beta}{3}} \mp \beta\sqrt{\frac{\beta}{3}} - \gamma = 0, \quad \frac{4}{27}\beta^3 = \gamma^2.$$

The pair has the Kronecker type $\mu_1^2 \mu_2 \dots \mu_{t+1}$ if $(4/27)\beta^3 = \gamma^2$ and $\mu_1 \mu_2 \dots \mu_{t+2}$ if $(4/27)\beta^3 \neq \gamma^2$. We have the bifurcation diagram (viii).

9) Let $\mathcal{A}(0, 0) = \lambda_1^2 \lambda_2^2 \lambda_3 \dots \lambda_t$. Consider the case $\lambda_1 \neq \infty$ and $\lambda_2 \neq \infty$. Then $\mathcal{A}(\beta, \gamma)$ has direct summands

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 1 \\ \beta & \lambda_1 \end{bmatrix} \right) \quad \text{and} \quad \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \lambda_2 & 1 \\ \gamma & \lambda_2 \end{bmatrix} \right).$$

Similar to the case 5) of the proof of Theorem 3.2, the pair has the Kronecker type $\mu_1^2 \mu_2 \dots \mu_{t+1}$ if $\beta = 0$ or $\gamma = 0$ and the Kronecker type $\mu_1 \mu_2 \dots \mu_{t+2}$ if $\beta \neq 0$, and $\gamma \neq 0$. We have the bifurcation diagram (ix). \square

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